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Full faithfulness theorem for torsion crystalline representations II

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Abstract

Let p be a prime number and $r \geq 0$ an integer. Let K be a complete discrete valuation field of mixed characteristic $(0,p)$ with perfect residue field and absolute ramification index e. In a previous paper, under the condition $er < p-1$, we showed a full faithfulness theorem for torsion crystalline representations of $\mathrm{Gal}(\overline{K}/K)$ with prescribed Hodge-Tate weights in $[0,r]$. In this paper, we prove the same theorem under the refined condition $e(r-1) < p-1$. This refined condition is the best possible for many K. We apply our full faithfulness theorem to show the non-existence of crystalline lifts with prescribed small Hodge-Tate weights for torsion representations associated with Tate curves.

1 Introduction

Let p be a prime number and $r \geq 0$ an integer. Let K be a complete discrete valuation field of mixed characteristic $(0,p)$ with perfect residue field and absolute ramification index e. Let $\pi = \pi_0$ be a uniformizer of K and π_n a p^n-th root of π such that $\pi_{n+1}^p = \pi_n$ for all $n \geq 0$. Put $K_\infty = \bigcup_{n \geq 0} K(\pi_n)$ and denote by G_K and G_∞ absolute Galois groups of K and K_∞, respectively. A torsion $\mathbb{Z}_p$-representation of G_K is said to be *torsion crystalline with Hodge-Tate weights in* $[0,r]$ if it can be written as the quotient of two lattices in some crystalline $\mathbb{Q}_p$-representation of G_K with Hodge-Tate weights in $[0,r]$. For example, a torsion $\mathbb{Z}_p$-representation of G_K is finite flat if and only if it is torsion crystalline with Hodge-Tate weights in $[0,1]$.

Our main result in this paper is the following.

Theorem 1. *Suppose $e(r-1) < p-1$. Then the functor from torsion crystalline $\mathbb{Z}_p$-representations of G_K with Hodge-Tate weights in $[0,r]$ to torsion $\mathbb{Z}_p$-representations of G_∞, obtained by restricting the action of G_K to G_∞, is fully faithful.*

This full faithfulness theorem was first proved by Breuil for $e = 1$ and $r < p-1$ via the Fontaine-Laffaille theory ([Br1], the proof of Théorèm 5.2). He also proved Theorem 1 under the assumptions $p > 2$ and $r \leq 1$ as a result on a study of the category of finite flat group schemes ([Br2, Theorem 3.4.3]). His result was extended to the case $p = 2$ in [Ki], [La], [Li2] (proved independently). On the other hand, Abrashkin proved the full faithfulness in the case where $p > 2, r < p$ and K is a finite unramified extension of $\mathbb{Q}_p$ ([Ab2, Section 8.3.3]). His proof is based on calculations of ramification bounds for torsion crystalline representations. In [Oz], a proof of Theorem 1 under the assumption $er < p-1$ is given via $(\varphi, \hat{G})$-modules. Our main result is a direct generalization of these results. We prove our main result by improving the methods of [Oz], thus main tools we use are $(\varphi, \hat{G})$-modules.

If K is a finite extension of $\mathbb{Q}_p$ and $e(r-1) \geq p-1$, there exist a lot of counter examples of the full faithfulness. In fact, denoting by G_1 the absolute Galois group of $K(\pi_1)$, we have

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Theorem 2 (= Special case of Corollary 26). *Suppose that K is a finite extension of $\mathbb{Q}_p$, and also suppose $e \mid (p-1)$ or $(p-1) \mid e$. If $e(r-1) \geq p-1$, then the functor from torsion crystalline $\mathbb{Z}_p$-representations of G_K with Hodge-Tate weights in $[0,r]$ to torsion $\mathbb{Z}_p$-representations of G_1, obtained by restricting the action of G_K to G_1, is not full.*

In particular, if $p=2$, then the full faithfulness does not hold for any finite extension K of $\mathbb{Q}_2$ and any $r \geq 2$. Theorem 2 implies that the condition "$e(r-1) < p-1$" in Theorem 1 is the best possible for our full faithfulness theorem for many finite extensions K of $\mathbb{Q}_p$.

The latter half part of this paper is deduced to a study of crystalline lifts. Motivated by [CL2, Question 5.5], for a given torsion representation T, we calculate the minimum integer r with the property that T has a crystalline lift with Hodge-Tate weights in $[0,r]$. We mainly compute such r for various $\mathbb{F}_p$-representations of rank 2. We apply Theorem 1 to guarantee the non-existence of crystalline lifts with "small" Hodge-Tate weights for torsion representations associated with Tate curves (see Proposition 22).

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2 Preliminaries

In this section and the next section, we fix an odd prime number $p > 2$. (We remark that, as explained in Introduction, Theorem 1 has been known if $p=2$. Thus we may omit this case for our proof of Theorem 1.) Let k be a perfect field of characteristic p, $W(k)$ the ring of Witt vectors with coefficients in k, $K_0 = W(k)[1/p]$, K a finite totally ramified extension of K_0, $\overline{K}$ a fixed algebraic closure of K and $G_K = \mathrm{Gal}(\overline{K}/K)$. We fix a uniformizer $\pi \in K$ and denote by $E(u)$ its Eisenstein polynomial over K_0. For any integer $n \geq 0$, let $\pi_n \in \overline{K}$ be a p^n-th root of π such that $\pi_{n+1}^p = \pi_n$. Let $K_\infty = \bigcup_{n\geq 0} K(\pi_n)$ and $G_\infty = \mathrm{Gal}(\overline{K}/K_\infty)$.

For any topological group H, we denote by $\mathrm{Rep}_{\mathrm{tor}}(H)$ the category of finite torsion $\mathbb{Z}_p$-modules equipped with continuous linear H-action. For an integer $r \geq 0$, we say that $T \in \mathrm{Rep}_{\mathrm{tor}}(G_K)$ is *torsion crystalline with Hodge-Tate weights in* $[0,r]$ if it can be written as the quotient of two lattices in some crystalline $\mathbb{Q}_p$-representation of G_K with Hodge-Tate weights in $[0,r]$. We denote by $\mathrm{Rep}^r_{\mathrm{tor}}(G_K)$ the category of them.

Let $R = \varprojlim \mathcal{O}_{\overline{K}}/p$, where $\mathcal{O}_{\overline{K}}$ is the integer ring of $\overline{K}$ and the transition maps are given by the p-th power map. Write $\underline{\pi} = (\pi_n)_{n\geq 0} \in R$ and let $[\underline{\pi}] \in W(R)$ be the Teichmüller representative of $\underline{\pi}$. Put $\mathfrak{S} = W(k)[\![u]\!]$. We equip a Frobenius endomorphism φ of $\mathfrak{S}$ by $u \mapsto u^p$ and the Frobenius on $W(k)$. We embed the $W(k)$-algebra $W(k)[u]$ into $W(R)$ via the map $u \mapsto [\underline{\pi}]$. This embedding extends to an embedding $\mathfrak{S} \hookrightarrow W(R)$. It is readily seen that the embedding $\mathfrak{S} \hookrightarrow W(R)$ is compatible with the Frobenius endomorphisms.

2.1. Kisin modules. A *φ-module* (*over* $\mathfrak{S}$) is an $\mathfrak{S}$-module $\mathfrak{M}$ equipped with a φ-semilinear map $\varphi\colon \mathfrak{M} \to \mathfrak{M}$. A morphism between two φ-modules $(\mathfrak{M}_1, \varphi_1)$ and $(\mathfrak{M}_2, \varphi_2)$ is an $\mathfrak{S}$-linear map $\mathfrak{M}_1 \to \mathfrak{M}_2$ compatible with φ_1 and φ_2. Denote by $'\mathrm{Mod}^r_{/\mathfrak{S}}$ the category of φ-modules $(\mathfrak{M}, \varphi)$ *of height* $\leq r$ in the sense that $\mathfrak{M}$ is of finite type over $\mathfrak{S}$ and the cokernel of $1 \otimes \varphi\colon \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \to \mathfrak{M}$ is killed by $E(u)^r$. Let $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ be the full subcategory of $'\mathrm{Mod}^r_{/\mathfrak{S}}$ consisting of finite $\mathfrak{S}$-modules which are killed by some power of p and have projective dimension 1 in the sense that $\mathfrak{M}$ has a two term resolution by finite free $\mathfrak{S}$-modules. We call an object of $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ a *torsion Kisin module* (*of height* $\leq r$). For any torsion Kisin module $\mathfrak{M}$, we define a torsion $\mathbb{Z}_p$-representation $T_{\mathfrak{S}}(\mathfrak{M})$ of G_∞ by

$$T_{\mathfrak{S}}(\mathfrak{M}) = \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} \mathfrak{S}^{\mathrm{ur}}).$$

Here, a G_∞-action on $T_{\mathfrak{S}}(\mathfrak{M})$ is given by $(\sigma.f)(x) = \sigma(f(x))$ for $\sigma \in G_\infty, f \in T_{\mathfrak{S}}(\mathfrak{M}), x \in \mathfrak{M}$.

2.2. $(\varphi, \hat{G})$*-modules.* We recall the theory of torsion $(\varphi, \hat{G})$-modules. Let S be the p-adic completion of the divided power envelope of $W(k)[u]$ with respect to the ideal generated by $E(u)$. There exists a unique Frobenius map $\varphi\colon S \to S$ defined by $\varphi(u) = u^p$. Put $S_{K_0} = S[1/p] = K_0 \otimes_{W(k)} S$. The inclusion $W(k)[u] \hookrightarrow W(R)$ via the map $u \mapsto [\underline{\pi}]$ induces φ-compatible inclusions $\mathfrak{S} \hookrightarrow S \hookrightarrow A_{\mathrm{cris}}$ and $S_{K_0} \hookrightarrow B^+_{\mathrm{cris}}$. Fix a choice of primitive p^i-root of unity ζ_{p^i} for $i \geq 0$ such that $\zeta^p_{p^{i+1}} = \zeta_{p^i}$. Put $\underline{\varepsilon} = (\zeta_{p^i})_{i\geq 0} \in R^\times$ and $t = \log([\underline{\varepsilon}]) \in A_{\mathrm{cris}}$. Denote by $\nu\colon W(R) \to W(\overline{k})$ a unique lift of the projection $R \to \overline{k}$, which extends to a map $\nu\colon B^+_{\mathrm{cris}} \to W(\overline{k})[1/p]$. For any subring $A \subset B^+_{\mathrm{cris}}$, we put $I_+A = \mathrm{Ker}(\nu \text{ on } B^+_{\mathrm{cris}}) \cap A$. For any integer $n \geq 0$, let $t^{\{n\}} = t^{r(n)}\gamma_{\tilde{q}(n)}(\frac{t^{p-1}}{p})$ where $n = (p-1)\tilde{q}(n) + r(n)$ with $\tilde{q}(n) \geq 0$, $0 \leq r(n) < p-1$ and $\gamma_i(x) = \frac{x^i}{i!}$ is the standard divided power. We define a subring $\mathcal{R}_{K_0}$ of B^+_{cris} as below:

$$\mathcal{R}_{K_0} = \{\sum_{i=0}^{\infty} f_i t^{\{i\}} \mid f_i \in S_{K_0} \text{ and } f_i \to 0 \text{ as } i \to \infty\}.$$

Put $\widehat{\mathcal{R}} = \mathcal{R}_{K_0} \cap W(R)$ and $I_+ = I_+\widehat{\mathcal{R}}$. Put $\hat{K} = \bigcup_{n\geq 0} K_\infty(\zeta_{p^n})$ and $\hat{G} = \mathrm{Gal}(\hat{K}/K)$. Lemma 2.2.1 in [Li1] shows that $\widehat{\mathcal{R}}$ (resp. $\mathcal{R}_{K_0}$) is a φ-stable $\mathfrak{S}$-algebra as a subring in $W(R)$ (resp. B^+_{cris}), and ν induces $\mathcal{R}_{K_0}/I_+\mathcal{R}_{K_0} \simeq K_0$ and $\widehat{\mathcal{R}}/I_+ \simeq S/I_+S \simeq \mathfrak{S}/I_+\mathfrak{S} \simeq W(k)$. Furthermore, $\widehat{\mathcal{R}}, I_+, \mathcal{R}_{K_0}$ and $I_+\mathcal{R}_{K_0}$ are G_K-stable, and G_K-actions on them factors through $\hat{G}$. For any torsion Kisin module $\mathfrak{M}$, we equip $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ with a Frobenius by $\varphi_{\widehat{\mathcal{R}}} \otimes \varphi_{\mathfrak{M}}$. It is known that the natural map $\mathfrak{M} \to \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ given by $x \mapsto 1 \otimes x$ is an injection ([CL2, Section 3.1]). By this injection, we regard $\mathfrak{M}$ as a $\varphi(\mathfrak{S})$-stable submodule of $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$.

Definition 3. A *torsion* $(\varphi, \hat{G})$*-module* (*of height* $\leq r$) is a triple $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi_{\mathfrak{M}}, \hat{G})$ where

1. $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ is a torsion Kisin module (of height $\leq r$),
2. $\hat{G}$ is an $\widehat{\mathcal{R}}$-semilinear $\hat{G}$-action on $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$,
3. the $\hat{G}$-action commutes with $\varphi_{\widehat{\mathcal{R}}} \otimes \varphi_{\mathfrak{M}}$,
4. $\mathfrak{M} \subset (\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M})^{H_K}$ where $H_K = \mathrm{Gal}(\hat{K}/K_\infty)$,
5. $\hat{G}$ acts on the $W(k)$-module $(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M})/I_+(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M})$ trivially.

A morphism between two torsion $(\varphi, \hat{G})$-modules $\hat{\mathfrak{M}}_1 = (\mathfrak{M}_1, \varphi_1, \hat{G})$ and $\hat{\mathfrak{M}}_2 = (\mathfrak{M}_2, \varphi_2, \hat{G})$ is a morphism $f\colon \mathfrak{M}_1 \to \mathfrak{M}_2$ of φ-modules such that $\widehat{\mathcal{R}} \otimes f\colon \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}_1 \to \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}_2$ is $\hat{G}$-equivariant. We denote by $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ the category of torsion $(\varphi, \hat{G})$-modules of height $\leq r$. We often regard $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ as a G_K-module via the projection $G_K \twoheadrightarrow \hat{G}$. For a $(\varphi, \hat{G})$-module $\hat{\mathfrak{M}}$, we define a $\mathbb{Z}_p$-representation $\hat{T}(\hat{\mathfrak{M}})$ of G_K by

$$\hat{T}(\hat{\mathfrak{M}}) = \mathrm{Hom}_{\widehat{\mathcal{R}},\varphi}(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}, \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} W(R)).$$

Here, G_K acts on $\hat{T}(\hat{\mathfrak{M}})$ by $(\sigma.f)(x) = \sigma(f(\sigma^{-1}(x)))$ for $\sigma \in G_K$, $f \in \hat{T}(\hat{\mathfrak{M}})$, $x \in \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$. Then, there exists a natural G_∞-equivariant map

$$\theta\colon T_{\mathfrak{S}}(\mathfrak{M}) \to \hat{T}(\hat{\mathfrak{M}})$$

defined by $\theta(f)(a \otimes m) = a\varphi(f(m))$ for $f \in T_{\mathfrak{S}}(\mathfrak{M})$, $a \in \widehat{\mathcal{R}}, m \in \mathfrak{M}$.

Fix a topological generator τ of $\mathrm{Gal}(\hat{K}/K_{p^\infty})$ where $K_{p^\infty} = \bigcup_{n\geq 0} K(\zeta_{p^n})$. We may suppose that $\zeta_{p^n} = \tau(\pi_n)/\pi_n$ for all n, and this implies $\tau(u) = [\underline{\varepsilon}]u$ in $W(R)$. There exists $\mathfrak{t} \in W(R) \smallsetminus pW(R)$ such that $\varphi(\mathfrak{t}) = pE(0)^{-1}E(u)\mathfrak{t}$. Such $\mathfrak{t}$ is unique up to units of $\mathbb{Z}_p$ (cf. [Li1, Example 2.3.5]).

Now we define the full subcategory $\mathrm{Mod}^{r,\hat{G},\mathrm{cris}}_{/\mathfrak{S}_\infty}$ of $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ consisting of objects $\hat{\mathfrak{M}}$ which satisfy the following condition; $\tau(x) - x \in u^p\varphi(\mathfrak{t})(W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M})$ for any $x \in \mathfrak{M}$.

Theorem 4. (1) ([CL2, Theorem 3.1.3 (1)]) *The map θ is an isomorphism of $\mathbb{Z}_p[G_\infty]$-modules.*
(2) ([GLS, Proposition 5.9]) *For any $T \in \mathrm{Rep}^r_{\mathrm{tor}}(G_K)$, there exists $\hat{\mathfrak{M}} \in \mathrm{Mod}^{r,\hat{G},\mathrm{cris}}_{/\mathfrak{S}_\infty}$ such that $T \simeq \hat{T}(\hat{\mathfrak{M}})$ of $\mathbb{Z}_p[G_K]$-modules.*
(3) ([Oz, Lemma 7]) *Suppose $e(r-1) < p-1$. Then the forgetful functor $\mathrm{Mod}^{r,\hat{G},\mathrm{cris}}_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ is fully faithful.*

2.3. Maximal Kisin modules. We give a very rough sketch of the theory of maximal models for Kisin modules. Our sketch here is the case where "$r = \infty$" in [CL1]. For any $\mathfrak{M} \in \mathrm{Mod}^r_{/\mathfrak{S}_\infty}$, put $\mathfrak{M}[1/u] = \mathfrak{S}[1/u] \otimes_{\mathfrak{S}} \mathfrak{M}$ and denote by $F_{\mathfrak{S}}(\mathfrak{M}[1/u])$ the (partially) ordered set (by inclusion) of torsion Kisin modules $\mathfrak{N}$ of finite height which is contained in $\mathfrak{M}[1/u]$ and $\mathfrak{N}[1/u] = \mathfrak{M}[1/u]$ as φ-modules. Here, we say that a torsion Kisin module is *of finite height* if it is of height $\leq s$ for some integer $s \geq 0$. The set $F_{\mathfrak{S}}(\mathfrak{M}[1/u])$ has a greatest element (cf. *loc. cit.*, Corollary 3.2.6). We denote this element by $\mathrm{Max}(\mathfrak{M})$. We say that $\mathfrak{M}$ is *maximal* if it is the greatest element of $F_{\mathfrak{S}}(\mathfrak{M}[1/u])$. The implication $\mathfrak{M} \mapsto \mathrm{Max}(\mathfrak{M})$ defines a functor "Max" from the category of torsion Kisin modules of finite height into the category $\mathrm{Max}_{/\mathfrak{S}_\infty}$ of maximal torsion Kisin modules. The category $\mathrm{Max}_{/\mathfrak{S}_\infty}$ is abelian (cf. *loc. cit.*, Theorem 3.3.8). Furthermore, the functor $T_{\mathfrak{S}} \colon \mathrm{Max}_{/\mathfrak{S}_\infty} \to \mathrm{Rep}_{\mathrm{tor}}(G_\infty)$, defined by $T_{\mathfrak{S}}(\mathfrak{M}) = \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} W(R))$, is exact and fully faithful (cf. *loc. cit.*, Corollary 3.3.10). It is not difficult to check that $T_{\mathfrak{S}}(\mathrm{Max}(\mathfrak{M}))$ is canonically isomorphic to $T_{\mathfrak{S}}(\mathfrak{M})$ as representations of G_∞ for any torsion Kisin module $\mathfrak{M}$.

Definition 5 ([CL1, Section 3.6.1])**.** Let d be a positive integer. Let $\mathfrak{n} = (n_i)_{i \in \mathbb{Z}/d\mathbb{Z}}$ be a sequence of non-negative integers of smallest period d. We define a torsion Kisin module $\mathfrak{M}(\mathfrak{n})$ as below:

- as a $k[\![u]\!]$-module, $\mathfrak{M}(\mathfrak{n}) = \bigoplus_{i \in \mathbb{Z}/d\mathbb{Z}} k[\![u]\!] e_i$;
- for all $i \in \mathbb{Z}/d\mathbb{Z}$, $\varphi(e_i) = u^{n_i} e_{i+1}$.

We denote by $\mathcal{S}_{\max}$ the set of sequences $\mathfrak{n} = (n_i)_{i \in \mathbb{Z}/d\mathbb{Z}}$ of integers $0 \leq n_i \leq p-1$ with smallest period d for some integer d except the constant sequence with value $p-1$. It is known that $\mathfrak{M}(\mathfrak{n})$ is maximal for any $\mathfrak{n} \in \mathcal{S}_{\max}$ ([CL1, Proposition 3.6.7]). If k is algebraically closed, then $\mathfrak{M}(\mathfrak{n})$ is simple in $\mathrm{Max}_{/\mathfrak{S}_\infty}$ for any $\mathfrak{n} \in \mathcal{S}_{\max}$ (cf. *loc. cit.*, Proposition 3.6.12) and furthermore, the converse holds; any simple object in $\mathrm{Max}_{/\mathfrak{S}_\infty}$ is of the form $\mathfrak{M}(\mathfrak{n})$ for some $\mathfrak{n} \in \mathcal{S}_{\max}$ (cf. *loc. cit.*, Proposition 3.6.8 and 3.6.12).

Proposition 6. *Assume that k is algebraically closed. Then any d-dimensional $\mathbb{F}_p$-representation T of G_∞ is of the form $T_{\mathfrak{S}}(\mathfrak{M}(\mathfrak{n}))$ for some $\mathfrak{n} \in \mathcal{S}_{\max}$ of period d.*

Proof. By Proposition 5.6 of [CL2], we see that T is of the form $T_{\mathfrak{S}}(\mathfrak{M})$ for some torsion Kisin module $\mathfrak{M}$. Replacing $\mathfrak{M}$ with $\mathrm{Max}(\mathfrak{M})$, we may suppose that $\mathfrak{M}$ is maximal. Since T is irreducible and the functor $T_{\mathfrak{S}}$ on $\mathrm{Max}_{/\mathfrak{S}_\infty}$ is exact and fully faithful, $\mathfrak{M}$ is a simple object of $\mathrm{Max}_{/\mathfrak{S}_\infty}$. By Proposition 3.6.8 and 3.6.12 of *loc. cit.*, we finish a proof. $\square$

3 Proof of Theorem 1

In this section, we prove Theorem 1. We start with the following lemma, which should be well-known to experts, but we include a proof here for the sake of completeness.

Lemma 7. *The full subcategory $\mathrm{Rep}^r_{\mathrm{tor}}(G_K)$ of $\mathrm{Rep}_{\mathrm{tor}}(G_K)$ is stable under formation of subquotients, direct sums and the association $T \mapsto T^\vee(r)$. Here $T^\vee = \mathrm{Hom}_{\mathbb{Z}_p}(T, \mathbb{Q}_p/\mathbb{Z}_p)$ is the dual representation of T.*

Proof. Let $T \in \mathrm{Rep}^r_{\mathrm{tor}}(G_K)$ be killed by p^n for some $n > 0$. Assertions for quotients and direct sums are clear. We prove that $T^\vee(r)$ is contained in $\mathrm{Rep}^r_{\mathrm{tor}}(G_K)$. There exist lattices $L_1 \subset L_2$ in some

crystalline $\mathbb{Q}_p$-representation of G_K and an exact sequence $0 \to L_1 \to L_2 \to T \to 0$ of $\mathbb{Z}_p[G_K]$-modules. This exact sequence induces an exact sequence $0 \to T \to L_1/p^nL_1 \to L_2/p^nL_2 \to T \to 0$ of finite $\mathbb{Z}_p[G_K]$-modules. By duality, we obtain an exact sequence $0 \to T^\vee \to (L_2/p^nL_2)^\vee \to (L_1/p^nL_1)^\vee \to T^\vee \to 0$ of finite $\mathbb{Z}_p[G_K]$-modules. Then we obtain a G_K-equivariant surjection $L_1^\vee \twoheadrightarrow T^\vee$ by the composite $L_1^\vee \twoheadrightarrow L_1^\vee/p^nL_1^\vee \overset{\sim}{\to} (L_1/p^nL_1)^\vee \twoheadrightarrow T^\vee$ of natural maps (here, for any free $\mathbb{Z}_p$-representation L of G_K, $L^\vee := \mathrm{Hom}_{\mathbb{Z}_p}(L, \mathbb{Z}_p)$ stands for the dual of L). Therefore, we obtain $L_1^\vee(r) \twoheadrightarrow T^\vee(r)$ and thus $T^\vee(r) \in \mathrm{Rep}^r_{\mathrm{tor}}(G_K)$. In below, we prove the stability assertion for subobjects. Let T' be a G_K-stable submodule of T. We have a G_K-equivariant surjection $f\colon L_1^\vee \twoheadrightarrow T^\vee \twoheadrightarrow (T')^\vee$. Let L_2' be a free $\mathbb{Z}_p$-representation of G_K such that its dual is the kernel of f. We have an exact sequence $0 \to (L_2')^\vee \to L_1^\vee \overset{f}{\to} (T')^\vee \to 0$ of $\mathbb{Z}_p[G_K]$-modules. Repeating the construction of the surjection $L_1^\vee \twoheadrightarrow T^\vee$, we obtain a G_K-equivariant surjection $L_2' = (L_2')^{\vee\vee} \twoheadrightarrow (T')^{\vee\vee} = T'$ and thus we have $T' \in \mathrm{Rep}^r_{\mathrm{tor}}(G_K)$. $\square$

Lemma 8. *Let d be a positive integer. Let $\mathfrak{n} = (n_i)_{i\in\mathbb{Z}/d\mathbb{Z}}$ be a sequence of non-negative integers of smallest period d. Suppose $\mathfrak{M}(\mathfrak{n})$ is of height $\leq r$. Then $\mathfrak{M}(\mathfrak{n})$ has a structure of an object of $\mathrm{Mod}^{r,\hat{G},\mathrm{cris}}_{/\mathfrak{S}_\infty}$.*

Proof. Choose any (p^d-1)-th root $\eta \in R$ of $\underline{\varepsilon}$. Since $[\eta]\cdot \exp(-t/(p^d-1))$ is a (p^d-1)-th root of unity, it is of the form $[a]$ for some $a \in \mathbb{F}_{p^d}^\times$. Replacing η with ηa^{-1}, we obtain $[\eta] = \exp(t/(p^d-1)) \in \widehat{\mathcal{R}}^\times$. Put $x_i = [\eta]^{m_i} \in \widehat{\mathcal{R}}^\times$ and $\bar{x}_i = \eta^{m_i} \in (\widehat{\mathcal{R}}/p\widehat{\mathcal{R}})^\times \subset R^\times$ for any $i \in \mathbb{Z}/d\mathbb{Z}$, where $m_i = \sum_{j=0}^{d-1} n_{i+j}p^{d-j}$. We know that $x_i - 1$ is contained in $I_+\widehat{\mathcal{R}}$ since $\nu([\eta]) = 1$. Now we define a $\hat{G}$-action on $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ by $\tau(e_i) := x_ie_i$ for the basis $\{e_i\}_{i\in\mathbb{Z}/d\mathbb{Z}}$ of $\mathfrak{M}(\mathfrak{n})$ as in Definition 5. It is not difficult to check that $\mathfrak{M}(\mathfrak{n})$ has a structure of an object of $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ via this $\hat{G}$-action. It suffices to prove that this $(\varphi,\hat{G})$-module is in fact an object of $\mathrm{Mod}^{r,\hat{G},\mathrm{cris}}_{/\mathfrak{S}_\infty}$. Let v_R be the valuation of R normalized such that $v_R(u) = 1/e$ and define $\tilde{\mathfrak{t}} = \mathfrak{t} \bmod pW(R)$ an element of R. Note that we have $v_R(\underline{\varepsilon}-1) = p/(p-1)$ and $v_R(\tilde{\mathfrak{t}}) = 1/(p-1)$ (here, the latter equation follows from the relation $\varphi(\mathfrak{t}) = pE(0)^{-1}E(u)\mathfrak{t}$). Thus we see that $v_R(\bar{x}_i - 1) = p^{v_p(m_i)}\cdot p/(p-1) \geq p^2/(p-1) \geq p/e + p/(p-1) = v_R(u^p\varphi(\mathfrak{t}))$. This implies the desired result. $\square$

Corollary 9. *Put $r_0 = \max\{r \in \mathbb{Z}_{\geq 0}; e(r-1) < p-1\}$. If $\mathfrak{n} \in \mathcal{S}_{\max}$, then $\mathfrak{M}(\mathfrak{n})$ has a structure of an object of $\mathrm{Mod}^{r_0,\hat{G},\mathrm{cris}}_{/\mathfrak{S}_\infty}$, which is uniquely determined. We denote this $(\varphi,\hat{G})$-module by $\hat{\mathfrak{M}}(\mathfrak{n})$.*

Proof. Since $\mathfrak{M}(\mathfrak{n})$ is of height $\leq r_0$ for any $\mathfrak{n} \in \mathcal{S}_{\max}$, the result follows from Theorem 4 (3) and Lemma 8. $\square$

Lemma 10. *The functor from tamely ramified $\mathbb{Z}_p$-representations of G_K to $\mathbb{Z}_p$-representations of G_∞, obtained by restricting the action of G_K to G_∞, is fully faithful.*

Proof. The result follows from the fact that G_K is topologically generated by G_∞ and the wild inertia subgroup of G_K. $\square$

We remark that any semi-simple $\mathbb{F}_p$-representation of G_K is automatically tame.

The following proposition is a key of our proof of Theorem 1.

Proposition 11. *Let $T \in \mathrm{Rep}_{\mathrm{tor}}(G_K)$ and $T' \in \mathrm{Rep}^r_{\mathrm{tor}}(G_K)$. Suppose that T is tame, $pT = 0$ and $e(r-1) < p-1$. Then all G_∞-equivariant morphisms $T \to T'$ and $T' \to T$ are in fact G_K-equivariant.*

Proof. By duality, it is enough to show that any G_∞-equivariant morphism $T \to T'$ is in fact G_K-equivariant. Consider the following commutative diagram:

$$\begin{tikzcd} \mathrm{Hom}_{I_K}(T,T') \arrow[r, hook] & \mathrm{Hom}_{I_K\cap G_\infty}(T,T') \\ \mathrm{Hom}_{G_K}(T,T') \arrow[u, hook] \arrow[r, hook] & \mathrm{Hom}_{G_\infty}(T,T'). \arrow[u, hook] \end{tikzcd}$$

Let K^{ur} be the maximal unramified extension of K, $I_K := \mathrm{Gal}(\overline{K}/K^{\mathrm{ur}})$ the inertia subgroup of G_K and $(K^{\mathrm{ur}})_\infty := \bigcup_{n\geq 0} K^{\mathrm{ur}}(\pi_n)$. Note that G_K is topologically generated by I_K and G_∞, and note also that $I_K \cap G_\infty = \mathrm{Gal}(\overline{K}/(K^{\mathrm{ur}})_\infty)$. Hence the above diagram allows us to reduce a proof to the case where k is algebraically closed. In the rest of this proof, we assume that k is algebraically closed. Under this assumption, an $\mathbb{F}_p$-representation of G_K is tame if and only if it is semi-simple by Maschke's theorem. Thus we may also assume that T is irreducible. We claim that $T|_{G_\infty}$ is also irreducible. If not, there exists a non-zero irreducible sub $\mathbb{F}_p[G_\infty]$-module W of $T|_{G_\infty}$. Let K^{t} be the maximal tamely ramified extension of K and $I_p := \mathrm{Gal}(\overline{K}/K^{\mathrm{t}})$ the wild inertia subgroup of G_K. We see that $K^{\mathrm{t}} \cap K_\infty = K$. Since $G_\infty \cap I_p$ acts on W trivially, the G_∞-action on W extends to G_K via the composition map $G_K \twoheadrightarrow \mathrm{Gal}(K^{\mathrm{t}}/K) \simeq G_\infty/(G_\infty \cap I_p)$. Thus we can regard W as an irreducible $\mathbb{F}_p[G_K]$-module. By Lemma 10, we see that W is a sub $\mathbb{F}_p[G_K]$-module of T. This contradicts the irreducibility of T and the claim follows. By the claim and Proposition 6, $T|_{G_\infty}$ is of the form $T_{\mathfrak{S}}(\mathfrak{M}(\mathfrak{n}))$ for some $\mathfrak{n} \in \mathcal{S}_{\max}$. Let $\hat{\mathfrak{M}}(\mathfrak{n})$ be the $(\varphi, \hat{G})$-module as in Corollary 9. We recall that $T_{\mathfrak{S}}(\mathfrak{M}(\mathfrak{n}))$ is isomorphic to $\hat{T}(\hat{\mathfrak{M}}(\mathfrak{n}))|_{G_\infty}$ (see Theorem 4 (1)), and hence we have an isomorphism $T|_{G_\infty} \simeq \hat{T}(\hat{\mathfrak{M}}(\mathfrak{n}))|_{G_\infty}$. Applying Lemma 10 again, we obtain an isomorphism $T \simeq \hat{T}(\hat{\mathfrak{M}}(\mathfrak{n}))$ as representations of G_K. Now we put $r_0 = \max\{r' \in \mathbb{Z}_{\geq 0}; e(r'-1) < p-1\}$. By our assumption for r, we have $r \leq r_0$. By Theorem 4 (2), there exists $\hat{\mathfrak{M}}' = (\mathfrak{M}', \varphi, \hat{G}) \in \mathrm{Mod}^{r_0,\hat{G},\mathrm{cris}}_{/\mathfrak{S}_\infty}$ such that $T' \simeq \hat{T}(\hat{\mathfrak{M}}')$. Recall that $\hat{\mathfrak{M}}(\mathfrak{n})$ is also an object of $\mathrm{Mod}^{r_0,\hat{G},\mathrm{cris}}_{/\mathfrak{S}_\infty}$ and furthermore, its underlying Kisin module $\mathfrak{M}(\mathfrak{n})$ is maximal ([CL1, Proposition 3.6.7]). We consider the following commutative diagram:

$$\begin{tikzcd} \mathrm{Hom}_{G_K}(T,T') \arrow[rr, hook] & & \mathrm{Hom}_{G_\infty}(T,T') \\ \mathrm{Hom}(\hat{\mathfrak{M}}', \hat{\mathfrak{M}}(\mathfrak{n})) \arrow[u, "\hat{T}"] \arrow[r, "\text{forgetful}"] & \mathrm{Hom}(\mathfrak{M}', \mathfrak{M}(\mathfrak{n})) \arrow[r, "\mathrm{Max}"] & \mathrm{Hom}(\mathrm{Max}(\mathfrak{M}'), \mathfrak{M}(\mathfrak{n})). \arrow[u, "T_{\mathfrak{S}}"'] \end{tikzcd}$$

The first bottom horizontal arrow is bijective by Theorem 4 (3) and so is the second by an easy argument. Since the right vertical arrow is bijective, the top horizontal arrow must be bijective. $\square$

Proof of Theorem 1. The full faithfulness follows from a dévissage argument and Proposition 11 as below. Let T, T' be objects of $\mathrm{Rep}^r_{\mathrm{tor}}(G_K)$. Take any Jordan-Höllder sequence $0 = T_0 \subset T_1 \subset \cdots \subset T_n = T$ of T. By Proposition 11, if an exact sequence $0 \to T' \to V \to T_i/T_{i-1} \to 0$ in $\mathrm{Rep}^r_{\mathrm{tor}}(G_K)$ splits as representation of G_∞, then it splits as a sequence of representations of G_K. On the other hand, note that the category $\mathrm{Rep}^r_{\mathrm{tor}}(G_K)$ is stable under formation of subquotients and direct sums by Lemma 7. Thus it is an exact category in the sense of Quillen ([Qu, Section 2]). Hence short exact sequences in $\mathrm{Rep}^r_{\mathrm{tor}}(G_K)$ gives rise to exact sequences of Hom's and Ext's in the usual way. (This property holds for any exact category.) Therefore, comparing exact sequences of Hom's and Ext's arising from $0 \to T_{i-1} \to T_i \to T_i/T_{i-1} \to 0$ in the category $\mathrm{Rep}^r_{\mathrm{tor}}(G_K)$ with that in the category $\mathrm{Rep}_{\mathrm{tor}}(G_\infty)$, we obtain the following implication (here, use Proposition 11 again): if we have $\mathrm{Hom}_{G_K}(T_{i-1}, T') = \mathrm{Hom}_{G_\infty}(T_{i-1}, T')$, then it gives the equality $\mathrm{Hom}_{G_K}(T_i, T') = \mathrm{Hom}_{G_\infty}(T_i, T')$. Hence a dévissage argument works and the desired full faithfulness follows. $\square$

Corollary 12. *Suppose $e(r-1) < p-1$. Then the essential image of $\mathrm{Rep}_{\mathrm{tor}}(G_K) \to \mathrm{Rep}_{\mathrm{tor}}(G_\infty)$ is stable under formation of subobjects and quotients.*

Proof. It is enough to prove that any G_∞-stable submodule T' of $T \in \mathrm{Rep}^r_{\mathrm{tor}}(G_K)$ is in fact G_K-stable. We proceed by induction on $n > 0$ such that $p^n T = 0$. Suppose $n = 1$. Take

any sequence $T' = T_0 \subset T_1 \subset \cdots \subset T_m = T$ of torsion G_∞-stable submodules of T such that T_i/T_{i-1} is irreducible for any i. As explained in the proof of Proposition 11, the G_∞-action on T_m/T_{m-1} can be (uniquely) extended to G_K, and then Proposition 11 implies that the projection map $T = T_m \twoheadrightarrow T_m/T_{m-1}$ is G_K-equivariant. Hence $T_{m-1} \subset T_m = T$ is G_K-stable. Repeating this argument, we see that $T' \subset T$ is G_K-stable. For $n > 1$, we can apply the same argument of the latter half proof of Theorem 3.4.3 in [Br2]. □

The following result follows from the same method of the proof of Corollary 3.4.4 in [Br2].

Corollary 13. *Suppose $e(r-1) < p-1$. Let V be a crystalline $\mathbb{Q}_p$-representation of G_K with Hodge-Tate weights in $[0,r]$ and $T \subset V$ a $\mathbb{Z}_p$-lattice which is stable under G_∞. Then T is stable under G_K.*

4 Crystalline lifts and c-weights

We continue to use the same notation as in Section 2 except for that we may allow $p = 2$. We remark that a torsion $\mathbb{Z}_p$-representation of G_K is torsion crystalline with Hodge-Tate weights in $[0,r]$ if there exist a lattice L in some crystalline $\mathbb{Q}_p$-representation of G_K with Hodge-Tate weights in $[0,r]$ and a G_K-equivariant surjection $f\colon L \twoheadrightarrow T$. We call f a *crystalline lift* (*of* T) *of weight* $\le r$. Our interest in this section is to determine the minimum integer r (if it exists) such that T admits crystalline lifts of weight $\le r$. We call this minimum integer the *c-weight of* T and denote it by $w_c(T)$. If T does not have crystalline lifts of weight $\le r$ for any integer r, then we define the c-weight $w_c(T)$ of T to be ∞. Motivated by [CL2, Question 5.5], we raise the following question.

Question 14. *For a torsion $\mathbb{Z}_p$-representation T of G_K, is the c-weight $w_c(T)$ of T finite? Furthermore, can we calculate $w_c(T)$?*

4.1. General properties of c-weights. We study general properties of c-weights. At first, by ramification estimates, it is known that c-weights may have infinitely large values ([CL2, Theorem 5.4]); for any $c > 0$, there exists a torsion $\mathbb{Z}_p$-extension T of G_K with $w_c(T) > c$. In below, we mainly consider representations with "small" c-weights. If c-weights are "small", they are closely related with *tame inertia weights*. Now we recall the definition of tame inertia weights. Let I_K be the inertia subgroup of G_K. Let T be a d-dimensional irreducible $\mathbb{F}_p$-representation of I_K. Then T is isomorphic to

$$\mathbb{F}_{p^d}(\theta_{d,1}^{n_1} \cdots \theta_{d,d}^{n_d})$$

for one sequence of integers between 0 and $p-1$, periodic of period d. Here, $\theta_{d,1}, \ldots, \theta_{d,d}$ are the fundamental characters of level d. The integers $n_1/e, \ldots, n_d/e$ are called the tame inertia weights of T. For any $\mathbb{F}_p$-representation T of G_K, the tame inertia weights of T are the tame inertia weights of the Jordan-Hölder quotients of $T|_{I_K}$.

Let $\chi_p\colon G_K \to \mathbb{Z}_p^\times$ be the p-adic cyclotomic character and $\bar{\chi}_p\colon G_K \to \mathbb{F}_p^\times$ the mod p cyclotomic character. It is well-known that $\bar{\chi}_p|_{I_K} = \theta_1^e$ where $\theta_1\colon I_K \twoheadrightarrow \mathbb{F}_p^\times$ is the fundamental character of level 1. In particular, denoting by K^{ur} the maximal unramified extension of K, we have $[K^{\mathrm{ur}}(\mu_p) : K^{\mathrm{ur}}] = (p-1)/\gcd(e, p-1)$.

Proposition 15. (1) *Minimum c-weights are invariant under finite unramified extensions of the base field K.*
(2) *The c-weight of an unramified torsion $\mathbb{Z}_p$-representation of G_K is* 0.
(3) *Put $\nu = (p-1)/\gcd(e, p-1)$. If $\nu(s-1) < w_c(T) \le \nu s$, then we have $\nu(s-1) < w_c(T^\vee) \le \nu s$. In particular, if $p-1 \mid e$, then we have $w_c(T) = w_c(T^\vee)$.*
(4) *Let T be an $\mathbb{F}_p$-representation of G_K and i the largest tame inertia weight of T. Then we have $w_c(T) \ge i$.*

Proof. (1) Let T be a torsion $\mathbb{Z}_p$-representations of G_K. Let K' be a finite unramified extension of K. It suffices to prove that T has crystalline lifts of weight $\le r$ if and only if $T|_{G_{K'}}$ has crystalline lifts of weight $\le r$. The "only if" assertion is clear and thus it is enough to prove the "if" assertion. Let $f\colon L \twoheadrightarrow T|_{G_{K'}}$ be a crystalline lift of $T|_{G_{K'}}$ of weight $\le r$. Since K'/K is unramified, $\mathrm{Ind}^{G_K}_{G_{K'}} L$ is a lattice in some crystalline $\mathbb{Q}_p$-representation of G_K with Hodge-Tate weights in $[0,r]$. Furthermore, the map

$$\mathrm{Ind}^{G_K}_{G_{K'}} L = \mathbb{Z}_p[G_K] \otimes_{\mathbb{Z}_p[G_{K'}]} L \to T, \quad \sigma \otimes x \mapsto \sigma(f(x))$$

is a G_K-equivariant surjection and hence we have done.
(2) The result follows from (1) immediately.
(3) Taking a finite unramified extension K' of K with the property $[K^{\mathrm{ur}}(\mu_p) : K^{\mathrm{ur}}] = [K'(\mu_p) : K']$, it follows from Lemma 7 that we have $\nu(s-1) < w_c(T|_{G'_K}) \le \nu s$ if and only if we have $\nu(s-1) < w_c((T^\vee)|_{G'_K}) \le \nu s$. Thus the result follows from the assertion (1).
(4) If $ew_c(T) \ge p-1$, then there is nothing to prove, and thus we may suppose that $ew_c(T) < p-1$. Let $L \twoheadrightarrow T$ be a crystalline lift of T of weight $\le w_c(T)$. Since the tame inertia polygon of L lies on the Hodge polygon of L ([CS, Théorème 1]), the largest slope of the former polygon is less than or equal to that of the latter polygon. This implies $w_c(T) \ge i$. □

Theorem 16. *Let T be a tamely ramified $\mathbb{F}_p$-representation of G_K. Let i be the largest tame inertia weight of T. Then we have $w_c(T) = \min\{h \in \mathbb{Z}_{\ge 0}; h \ge i\}$.*

Proof. The proof below is essentially due to Caruso and Liu [CL2, Theorem 5.7], but we give a proof here for the sake of completeness. Put $i_0 = \min\{h \in \mathbb{Z}_{\ge 0}; h \ge i\}$. By Proposition 15 (4), we have $w_c(T) \ge i_0$. Thus it suffices to show $w_c(T) \le i_0$. We note that $T|_{I_K}$ is semi-simple. Any irreducible component T_0 of $T|_{I_K}$ is of the form $\mathbb{F}_{p^d}(\theta_{d,1}^{n_1} \cdots \theta_{d,d}^{n_d})$ for one sequence of integers between 0 and $p-1$, periodic of period d. We decompose $n_j = em_j + n'_j$ by integers $0 \le m_j \le i_0$ and $0 \le n'_j < e$. Now we define an integer $k_{j,\ell}$ by

$$k_{j,\ell} := \begin{cases} e & \text{if } 1 \le \ell \le m_j, \\ n'_j & \text{if } \ell = m_j + 1, \\ 0 & \text{if } \ell > m_j + 1. \end{cases}$$

Note that we have $n_j = \sum_{\ell=1}^{i_0} k_{j,\ell}$, and also have an I_K-equivariant surjection

$$T_0 = \mathbb{F}_{p^d}(\theta_{d,1}^{n_1} \ldots \theta_{d,d}^{n_d}) = \bigotimes_{\ell=1,\ldots i_0, \mathbb{F}_{p^d}} \mathbb{F}_{p^d}(\theta_{d,1}^{k_{1,\ell}} \ldots \theta_{d,d}^{k_{d,\ell}}) \twoheadleftarrow \bigotimes_{\ell=1,\ldots i_0, \mathbb{F}_p} \mathbb{F}_{p^d}(\theta_{d,1}^{k_{1,\ell}} \ldots \theta_{d,d}^{k_{d,\ell}}).$$

By a classical result of Raynaud, each $\mathbb{F}_{p^d}(\theta_{d,1}^{k_{1,\ell}} \cdots \theta_{d,d}^{k_{d,\ell}})$ comes from a finite flat group scheme defined over K^{ur}. We should remark that such a finite flat group scheme is in fact defined over a finite unramified extension of K. Since any finite flat group scheme can be embedded in a p-divisible group, the above observation implies the following: there exist a finite unramified extension K' over K, a lattice L in some crystalline $\mathbb{Q}_p$-representation of $G_{K'}$ with Hodge-Tate weights in $[0,i_0]$ and an I_K-equivariant surjection $f\colon L \twoheadrightarrow T$. The map f induces an I_K-equivariant surjection $\tilde{f}\colon L/pL \twoheadrightarrow T$. Since L/pL and T is finite, we see that $\tilde{f}$ is in fact $G_{K''}$-equivariant for some finite unramified extension K'' over K', and then so is f. Therefore, we obtain $w_c(T|_{G_{K''}}) \le i_0$. By Proposition 15 (1), we obtain $w_c(T) \le i_0$. □

4.2. Rank 2 cases. We give some computations of c-weights related with torsion representations of rank 2. We prove the following lemma by an almost identical method with [GLS, Lemma 9.4].

Lemma 17. *Let K be a finite extension of $\mathbb{Q}_p$. Let E be a finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}$. Let i and ν be integers such that ν is divisible by $[K(\mu_p) : K]$. Suppose that T is an*

$\mathbb{F}$-representation of G_K which sits in an exact sequence $(*)\colon 0 \to \mathbb{F}(i) \to T \to \mathbb{F} \to 0$ of $\mathbb{F}$-representations of G_K. Then there exist a ramified degree at most 2 extension E' over E, with integer ring $\mathcal{O}_{E'}$, and an unramified continuous character $\chi\colon G_K \to \mathbb{F}^\times$ with trivial reduction such that $(*)$ is the reduction of some exact sequence $0 \to \mathcal{O}_{E'}(\chi\chi_p^{i+\nu}) \to \Lambda \to \mathcal{O}_{E'} \to 0$ of free $\mathcal{O}_{E'}$-representations of G_K. Furthermore, we have the followings:
(1) If $i+\nu = 1$ or $\bar{\chi}_p^{1-i} \neq 1$, then we can take $E' = E$ and $\chi = 1$.
(2) If $i+\nu = 0$ and T is unramified, then we can take $E' = E$, $\chi = 1$ and Λ to be unramified.

Proof. Suppose $i+\nu = 1$ (resp. $\bar{\chi}_p^{1-i} \neq 1$). Then the map $H^1(K, \mathcal{O}_E(i+\nu)) \to H^1(K, \mathbb{F}(i))$ arising from the exact sequence $0 \to \mathcal{O}_E(i+\nu) \xrightarrow{\varpi} \mathcal{O}_E(i+\nu) \to \mathbb{F}(i) \to 0$ is surjective since $H^2(K, \mathcal{O}_E(1)) \simeq \mathcal{O}_E$ (resp. $H^2(K, \mathcal{O}_E(i+\nu)) = 0$), where ϖ is a uniformizer of E. Hence we obtained a proof of (1). The assertion (2) follows immediately from the fact that the natural map $H^1(G_K/I_K, \mathcal{O}_E) \to H^1(G_K/I_K, \mathbb{F})$ is surjective.

In below, we always assume that $i+\nu \neq 1$ and $\bar{\chi}_p^{1-i} = 1$. Let $L \in H^1(K, \mathbb{F}(i))$ be a 1-cocycle corresponding to $(*)$. We may suppose $L \neq 0$. For any unramified continuous character $\chi\colon G_K \to \mathbb{F}^\times$ with trivial reduction, we denote by

$$\delta^1_\chi\colon H^1(K, \mathbb{F}(i)) \to H^2(K, \mathcal{O}_E(\chi\chi_p^{i+\nu}))$$
$$(\text{resp. } \delta^0_\chi\colon H^0(K, E/\mathcal{O}_E(\chi^{-1}\chi_p^{1-i-\nu})) \to H^1(K, \mathbb{F}))$$

the connection map arising from the exact sequence $0 \to \mathcal{O}_E(\chi\chi_p^{i+\nu}) \xrightarrow{\varpi} \mathcal{O}_E(\chi\chi_p^{i+\nu}) \to \mathbb{F}(i) \to 0$ (resp. $0 \to \mathbb{F} \to E/\mathcal{O}_E(\chi^{-1}\chi_p^{1-i-\nu}) \xrightarrow{\varpi} E/\mathcal{O}_E(\chi^{-1}\chi_p^{1-i-\nu}) \to 0$) of $\mathcal{O}_E[G_K]$-modules. Consider the following commutative diagram:

$$\begin{tikzcd}
H^1(K, \mathbb{F}(i)) \arrow[d, "\delta^1_\chi"'] & \times & H^1(K, \mathbb{F}) \arrow[rr] & & E/\mathcal{O}_E \arrow[d, equal] \\
H^2(K, \mathcal{O}_E(\chi\chi_p^{i+\nu})) & \times & H^0(K, E/\mathcal{O}_E(\chi^{-1}\chi_p^{1-i-\nu})) \arrow[u, "\delta^0_\chi"] \arrow[rr] & & E/\mathcal{O}_E
\end{tikzcd}$$

Since the above two pairings are perfect, we see that L lifts to $H^1(G_K, \mathcal{O}_E(\chi\chi_p^{i+\nu}))$ if and only if H is contained in the image of δ^0_χ. Here, $H \subset H^1(K, \mathbb{F})$ is the annihilator of L under the local Tate pairing $H^1(K, \mathbb{F}(i)) \times H^1(K, \mathbb{F}) \to E/\mathcal{O}_E$. Let $n \geq 1$ be the largest integer with the property that $\chi^{-1}\chi_p^{1-i-\nu} \equiv 1 \bmod \varpi^n$ (such n exists since $\bar{\chi}_p^{1-i} = 1$ and $1-i-\nu \neq 0$). We define $\alpha_\chi\colon G_K \to \mathcal{O}_E$ by the relation $\chi^{-1}\chi_p^{1-i-\nu} = 1 + \varpi^n\alpha_\chi$, and denote $(\alpha_\chi \bmod \varpi)\colon G_K \to \mathbb{F}$ by $\bar{\alpha}_\chi$. By definition, $\bar{\alpha}_\chi$ is a non-zero element of $H^1(K, \mathbb{F})$, and it is not difficult to check that the image of δ^0_χ is generated by $\bar{\alpha}_\chi$. If $\bar{\alpha}_\chi$ is contained in H for some χ, we are done. Suppose this is not the case.

Suppose that H is not contained in the unramified line in $H^1(K, \mathbb{F})$. We claim that we can choose χ such that $\bar{\alpha}_\chi$ is ramified. Let m be the largest integer with the property that $(\chi^{-1}\chi_p^{1-i-\nu})|_{I_K} \equiv 1 \bmod \varpi^n$. Clearly, we have $m \geq n$. If $m = n$, then we are done and thus we may assume $m > n$. Fix a lift $g \in G_K$ of the Frobenius of K. We see that $\bar{\alpha}_\chi(g) \neq 0$. Let χ' be the unramified character sending g to $1 + \varpi^n\alpha_\chi(g)$. Then χ' has trivial reduction. After replacing χ with $\chi\chi'$, we reduce the case where $m = n$ and thus the claim follows. Suppose $\bar{\alpha}_\chi$ is ramified. Then there exists a unique $\bar{x} \in \mathbb{F}^\times$ such that $\bar{\alpha}_\chi + u_{\bar{x}} \in H$ where $u_{\bar{x}}\colon G_K \to \mathbb{F}$ is the unramified character sending g to $\bar{x}$. Denote by χ'' the unramified character sending g to $1 + \varpi^n\alpha_\chi(g)$. Replacing χ with $\chi\chi''$, we have done.

Suppose that H is contained in the unramified line in $H^1(K, \mathbb{F})$ (thus H and the unramified line coincide with each other). By replacing E with $E(\sqrt{\varpi})$, we may assume that $n > 1$. Let χ_0 be a character defined by χ times the unramified character sending our fixed g to $1 + \varpi$. Since $n > 1$, we see that $\chi_0^{-1}\chi_p^{1-i-\nu} \equiv 1 \bmod \varpi$ and $\chi_0^{-1}\chi_p^{1-i-\nu} \not\equiv 1 \bmod \varpi^2$. We define $\alpha_{\chi_0}\colon G_K \to \mathcal{O}_E$ by the relation $\chi_0^{-1}\chi_p^{1-i-\nu} = 1 + \varpi\alpha_{\chi_0}$, and denote $(\alpha_{\chi_0} \bmod \varpi)\colon G_K \to \mathbb{F}$ by $\bar{\alpha}_{\chi_0}$. By definition and the assumption $n > 1$, $\bar{\alpha}_{\chi_0}$ is a non-zero unramified element of $H^1(K, \mathbb{F})$, hence it is contained in H. Therefore, we have done. $\square$

Let K be a finite extension of $\mathbb{Q}_p$, $n \geq 2$ an integer and $\chi\colon G_K \to E^\times$ an unramified character. Since any E-representation of G_K which is an extension of E by $E(\chi\chi_p^n)$ is automatically crystalline, we obtain the following.

Proposition 18. *Suppose $p > 2$. Let K be a finite unramified extension of $\mathbb{Q}_p$. Let $T \in \mathrm{Rep}_{\mathrm{tor}}(G_K)$ be killed by p and sit in an exact sequence $0 \to \mathbb{F}_p(i) \to T \to \mathbb{F}_p \to 0$ of $\mathbb{F}_p$-representations of G_K. Then we have the followings:*
(1) If $i = 0$ and T is unramified, then we have $w_c(T) = 0$.
(2) If $i = 0$ and T is not unramified, then we have $w_c(T) = p - 1$.
(3) If $i = 2, \dots, p-2$, then we have $w_c(T) = i$.

Proof. (1), (2) By Lemma 17 (2), it suffices to prove that T is not torsion crystalline with Hodge-Tate weights in $[0, p-2]$ if T is not unramified. Let K_T be the definition field of the representation T of G_K and put $G = \mathrm{Gal}(K_T/K)$. Let G^j be the upper numbering j-th ramification subgroup of G (in the sense of [Se]). Since T is not unramified and killed by p, we see that K_T is a totally ramified degree p extension over K. Thus G^1 is the wild inertia subgroup of G and $G^1 = G$, which does not act on T trivial by the definition of G. Thus we obtain the desired result by ramification estimates of [Fo] (or [Ab1]) for torsion crystalline representations with Hodge-Tate weights in $[0, p-2]$: if T is torsion crystalline with Hodge-Tate weights in $[0, p-2]$, then G^j acts on T trivial for any $j > (p-2)/(p-1)$.
(3) The result follows immediately from Proposition 15 (4) and Lemma 17. $\square$

Corollary 19. *Let K be a finite unramified extension of $\mathbb{Q}_p$. Then any 2-dimensional $\mathbb{F}_p$-representation of G_K is torsion crystalline with Hodge-Tate weights in $[0, 2p-2]$.*

Proof. If T is irreducible, the result follows from Theorem 16. Assume that T is reducible. Since K is unramified over $\mathbb{Q}_p$, any continuous character $G_K \to \mathbb{F}_p^\times$ is of the form $\chi\bar{\chi}_p^i$ for some unramified character χ and some integer i. Replacing K with its finite unramified extension, we may assume that T sits in an exact sequence $0 \to \mathbb{F}_p(i) \to T \to \mathbb{F}_p(j) \to 0$ of $\mathbb{F}_p$-representations of G_K, where i and j are integers in the range $[0, p-2]$ (we remark that $w_c(T)$ is invariant under unramified extensions of K by Proposition 15 (1)). It follows from Lemma 17 that $w_c(T(-j)) \leq p$. Therefore, we obtain $w_c(T) = w_c(T(-j) \otimes_{\mathbb{F}_p} \mathbb{F}_p(j)) \leq w_c(T(-j)) + w_c(\mathbb{F}_p(j)) \leq p + (p-2) = 2p-2$. $\square$

4.3. Extensions of $\mathbb{F}_p$ by $\mathbb{F}_p(1)$. By Lemma 17, we know that the c-weight $w_c(T)$ of an $\mathbb{F}_p$-representation T of G_K which sits in an exact sequence $0 \to \mathbb{F}_p(1) \to T \to \mathbb{F}_p \to 0$ of $\mathbb{F}_p$-representations of G_K, is less than or equal to p. In below, we calculate $w_c(T)$ for such T more precisely. We should remark that such T is written as p-torsion points of a Tate curve. Hence we consider torsion representations coming from Tate curves.

Let v_K be the valuation of K normalized such that $v_K(K^\times) = \mathbb{Z}$, and take any $q \in K^\times$ with $v_K(q) > 0$. Let E_q be the Tate curve over K associated with q and $E_q[p^n]$ the module of p^n-torsion points of E_q for any integer $n > 0$. It is well-known that there exists an exact sequence

$$(\#) \quad 0 \to \mu_{p^n} \to E_q[p^n] \to \mathbb{Z}/p^n\mathbb{Z} \to 0$$

of $\mathbb{Z}_p[G_K]$-modules. Here, μ_{p^n} is the group of p^n-th roots of unity in $\overline{K}$. Let $x_n\colon G_K \to \mu_{p^n}$ be the 1-cocycle defined to be the image of 1 for the connection map $H^0(K, \mathbb{Z}/p^n\mathbb{Z}) \to H^1(K, \mu_{p^n})$ arising from the exact sequence (#). Then x_n corresponds to $q \bmod (K^\times)^{p^n}$ via the isomorphism $K^\times/(K^\times)^{p^n} \simeq H^1(K, \mu_{p^n})$ of Kummer theory. Thus the exact sequence (#) splits if and only if $q \in (K^\times)^{p^n}$.

First we consider the case $p \mid v_K(q)$ (i.e. *peu ramifié* case).

Lemma 20. *Let K be a finite extension of $\mathbb{Q}_p$. If $p \mid v_K(q)$, then $E_q[p]$ is the reduction modulo p of a lattice in some 2-dimensional crystalline $\mathbb{Q}_p$-representation with Hodge-Tate weights in $[0, 1]$.*

Proof. Since $p \mid v_K(q)$, there exists $q' \in K^\times$ such that $v_K(q'-1) > 0$ and $q \equiv q' \bmod (K^\times)^p$. Considering the exact sequence $0 \to \mathbb{Z}_p(1) \to L \to \mathbb{Z} \to 0$ of $\mathbb{Z}_p$-representations of G_K corresponding to q' via the isomorphism $H^1(K, \mathbb{Z}_p(1)) \simeq \varprojlim_n K^\times/(K^\times)^{p^n}$ of Kummer theory, we obtain the desired result. □

Corollary 21. *Suppose that K is a finite extension of $\mathbb{Q}_p$, $(p-1) \nmid e$ and $p \mid v_K(q)$. Then we have $w_c(E_q[p]) = 1$.*

Proof. By the assumption $(p-1) \nmid e$, we know that the largest tame inertia weight of $E_q[p]$ is positive. Thus Proposition 15 (4) shows $w_c(E_q[p]) \geq 1$. The inequality $w_c(E_q[p]) \leq 1$ follows from Lemma 20. □

Next we consider the case $p \nmid v_K(q)$ (i.e. *très ramifié* case).

Proposition 22. *If $e(r-1) < p-1$ and $p \nmid v_K(q)$, then $E_q[p^n]$ is not torsion crystalline with Hodge-Tate weights in $[0, r]$ for any $n > 0$.*

Remark 23. If $e = 1$, the fact that $E_\pi[p^n]$ is not torsion crystalline with Hodge-Tate weights in $[0, p-1]$ immediately follows from the theory of ramification bound as below. We may suppose $n = 1$. Suppose $E_\pi[p]$ is torsion crystalline with Hodge-Tate weights in $[0, p-1]$. Then the upper numbering j-th ramification subgroup G_K^j of G_K (in the sense of [Se]) acts trivially on $E_\pi[p]$ for any $j > 1$ ([Ab1, Section 6, Theorem 3.1]). However, this contradicts the fact that the upper bound of the ramification of $E_\pi[p]$ is $1 + 1/(p-1)$.

Proof of Proposition 22. We may suppose $n = 1$. We choose any uniformizer π' of K. Putting $v_K(q) = m$, we can write $q = (\pi')^m x$ with some unit x of the integer ring of K. Since m is prime to p, we have a decomposition $x = \zeta_\ell y^m$ in $K^\times$ for some $\ell > 0$ prime to p and $y \in K$ with $v_K(y-1) > 0$. Here ζ_ℓ is a (not necessary primitive) ℓ-th root of unity. Since ℓ is prime to p, we have $\zeta_\ell = \zeta_\ell^{ps}$ for some integer s. We put $\pi = \pi' y$. This is a uniformizer of K. Choose any p-th root π_1 of π and put $q_1 = \zeta_\ell^s \pi_1^m \in K(\pi_1)^\times$. Then we have $q = q_1^p \in (K(\pi_1)^\times)^p$ and in particular, the exact sequence (#) (for $n = 1$) splits as representations of $\mathrm{Gal}(\overline{K}/K(\pi_1))$. Now assume that $E_q[p]$ is torsion crystalline with Hodge-Tate weights in $[0, r]$. Then (#) (for $n = 1$) splits as representations of G_K by Theorem 1. This contradicts the assumption $p \nmid v_K(q)$ (and hence $q \notin (K^\times)^p$). □

Now we put $r_1 = \min\{r \in \mathbb{Z}_{\geq 0}; e(r-1) \geq p-1\}$. Recall that we have $[K^{\mathrm{ur}}(\mu_p) : K^{\mathrm{ur}}] = (p-1)/\gcd(e, p-1)$.

Lemma 24. *Let K be a finite extension of $\mathbb{Q}_p$. Then $E_q[p]$ is torsion crystalline with Hodge-Tate weights in $[0, 1 + (p-1)/\gcd(e, p-1)]$.*

Proof. Taking a finite unramified extension K' of K such that $[K^{\mathrm{ur}}(\mu_p) : K^{\mathrm{ur}}] = [K'(\mu_p) : K']$, we obtain $w_c((E_q[p])|_{G_{K'}}) \leq 1 + (p-1)/\gcd(e, p-1)$ by Lemma 17. Thus we have $w_c(E_q[p]) \leq 1 + (p-1)/\gcd(e, p-1)$ by Proposition 15 (1). □

Corollary 25. *Suppose that K is a finite extension of $\mathbb{Q}_p$, and also suppose $e \mid (p-1)$ or $(p-1) \mid e$. We further suppose that $p \nmid v_K(q)$. Then we have $w_c(E_q[p]) = r_1$.*

Proof. We have $w_c(E_q[p]) \leq r_1$ by Lemma 24. In addition, we also have $w_c(E_q[p]) \geq r_1$ by Proposition 22. □

Lemma 24 gives some non-fullness results on torsion crystalline representations. We denote by G_1 the absolute Galois group of $K(\pi_1)$.

Corollary 26. *Suppose that K is a finite extension of $\mathbb{Q}_p$. If $r \geq 1 + (p-1)/\gcd(e, p-1)$, then the functor from torsion crystalline $\mathbb{Z}_p$-representations of G_K with Hodge-Tate weights in $[0, r]$ to torsion $\mathbb{Z}_p$-representations of G_1, obtained by restricting the action of G_K to G_1, is not full.*

Proof. Two representations $E_\pi[p]$ and $\mathbb{F}_p(1) \oplus \mathbb{F}_p$ are objects of $\mathrm{Rep}^r_{\mathrm{tor}}(G_K)$ by Lemma 24. They are not isomorphic as representations of G_K but isomorphic as representations of G_1. Thus the desired non-fullness follows. $\square$

Corollary 27. *Suppose that any one of the following holds:*

- $p = 2$ *and* K *is a finite extension of* $\mathbb{Q}_2$ *(in this case* $r_1 = 2$*);*
- K *is a finite unramified extension of* $\mathbb{Q}_p$ *(in this case* $r_1 = p$*);*
- K *is a finite extension of* $\mathbb{Q}_p(\mu_p)$ *(in this case* $r_1 = 2$*).*

Then the functor from torsion crystalline $\mathbb{Z}_p$*-representations of* G_K *with Hodge-Tate weights in* $[0, r_1]$ *to torsion* $\mathbb{Z}_p$*-representations of* G_1*, obtained by restricting the action of* G_K *to* G_1*, is not full.*

References

[Ab1] Victor Abrashkin, *Modular representations of the Galois group of a local field and a generalization of a conjecture of Shafarevich*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 6, 1135–1182 (Russian), Math. USSR-Izv. **35**, no. 3, 469–518 (English).

[Ab2] Victor Abrashkin, *Group schemes of period* $p > 2$, Proc. Lond. Math. Soc. (3), **101** (2010), 207–259.

[Br1] Christophe Breuil, *Une application du corps des normes*, Compos. Math. **117** (1999) 189–203.

[Br2] Christophe Breuil, *Integral p-adic Hodge theory*, in Algebraic geometry 2000, Azumino (Hotaka), Adv. Stud. Pure Math., vol. 36, Math. Soc. Japan, 2002, 51–80.

[CL1] Xavier Caruso and Tong Liu, *Quasi-semi-stable representations*, Bull. Soc. Math. France, **137** (2009), no. 2, 185–223.

[CL2] Xavier Caruso and Tong Liu, *Some bounds for ramification of* p^n*-torsion semi-stable representations*, J. Algebra, **325** (2011), 70–96.

[CS] Xavier Caruso and David Savitt, *Polygons de Hodge, de Newton et de línertie moderee des representations semi-stables*, Math. Ann. **343** (2009): 773–789.

[Fo] Jean-Marc Fontaine, *Scémas propres et lisses sur* $\mathbb{Z}$, Proceedings of the Indo-French Conference on Geometry Bombay, 1989, (Hindustan Book Agency 1993), 43–56.

[FL] Jean-Marc Fontaine and Guy Laffaille, *Construction de représentations p-adiques*, Ann. Sci. École Norm. Sup. **15** (1982), 547–608.

[GLS] Toby Gee, Tong Liu and David Savitt, *The Buzzard-Diamond-Jarvis conjecture for unitary groups*, preprint, arXiv:1203.2552v2

[Ki] Wansu Kim, *The classification of p-divisible groups over 2-adic discrete valuation rings*, Math. Res. Lett. **19** (2012), no. 1, 121–141.

[La] Eike Lau, *A relation between Dieudonné displays and crystalline Dieudonné theory*, preprint, arXiv:1006.2720v3

[Li1] Tong Liu, *A note on lattices in semi-stable representations*, Math. Ann. **346** (2010), 117–138.

[Li2] Tong Liu, *The correspondence between Barsotti-Tate groups and Kisin modules when $p = 2$*, preprint.

[Qu] Daniel Quillen, *Higher algebraic K-theory: I*, in Algebraic K-theory, I: Higher K-theories (Seattle, 1972), Lecture Notes in Math. **341**, Springer-Verlag, New York, 1973, 85–147.

[Se] Jean-Pierre Serre, *Corps locaux*, Hermann, Paris, 1968.

[Oz] Yoshiyasu Ozeki, *Full faithfulness theorem for torsion crystalline representations*, preprint, arXiv:1206.4751v3